Limite Fondamentale

(1) $\lim_{n \to \infty} \left(1+\frac{1}{n}\right)^n = e.$

(2) $\lim_{x \to 0} \frac{\ln(1+x)}{x} = 1$

$\boxed{x = \frac{1}{n}}$; $x \to 0$; $n \to \infty$

$$\lim_{n \to \infty} \frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}} = 1$$

$$\boxed{\lim_{n \to \infty} n \ln\left(1+\frac{1}{n}\right) = 1}$$

(3) $\lim_{x \to 0} \frac{e^x - 1}{x} = 1.$

(4) $\lim_{x \to 0} \frac{\sin x}{x} = 1.$

Exerciții - Limite

$\ln x - \ln y = \ln \frac{x}{y}$

① $f:\mathbb{R}\to\mathbb{R}$, $f(x) = 2x + \ln(x^2+x+1)$

$\lim\limits_{x\to+\infty}\left(f(x+1) - f(x)\right) = ?$ Notăm limita cu L.

Soluție: $f(x+1) - f(x) = \cancel{2x}+2+\ln\left((x+1)^2+x+2\right) - \cancel{2x} - \ln(x^2+x+1) =$

$$= 2+\ln\left(\frac{(x+1)^2+x+2}{x^2+x+1}\right) = 2+\ln\left(\frac{x^2+2x+1+x+2}{x^2+x+1}\right) = 2+\ln\left(\frac{x^2+3x+3}{x^2+x+1}\right)$$

Trecând la limită,

$$L = 2 + \lim_{x \to +\infty} \ln\left(\boxed{\frac{x^2+3x+3}{x^2+x+1}}\right) = 2 + \underbrace{\ln 1}_{=0} = \boxed{2}$$

$$\lim_{x \to +\infty} \frac{x^2+3x+3}{x^2+x+1} = \lim_{x \to +\infty} \frac{1 + \cancel{\frac{3}{x}}^{\to 0} + \cancel{\frac{3}{x^2}}^{\to 0}}{1 + \cancel{\frac{1}{x}}^{\to 0} + \cancel{\frac{1}{x^2}}^{\to 0}} = \frac{1}{1} = 1$$

$$\boxed{L = 2}$$

<u>Observatie</u>: $\dfrac{ax^n + \dots}{bx^n + \dots} \to$ prin factor comun fortat x^n,

$$\boxed{L = \frac{a}{b}}$$

② $f : \mathbb{R} \to (0, +\infty)$, $f(x) = 6^x - 3^x + 2^x = 1 - \frac{1}{2^x} + \frac{1}{3^x}$

$\lim_{x \to 0} \frac{\ln(f(x))}{x} = ?$

Notam $L \to \lim$

Soluție: $\ln(6^x - 3^x + 2^x)$

$\ln(6^x : 3^x \cdot 2^x) =$

$= \ln 6^x - \ln 3^x + \ln 2^x$

$\ln(6^x - 3^x + 2^x) \neq \frac{\ln(6^x) \ln(2^x)}{\ln(3^x)}$

$\ln(a+b) \neq \ln a \ln b$!!

$\ln(a \cdot b) = \ln a + \ln b$ ✓

Aplicăm Regula L'Hôpital, (caz $\left[\frac{0}{0}\right]$)

$\dfrac{\ln(f(x))}{\cancel{x}}$

$$L = \lim_{x \to 0} \frac{\frac{1}{f(x)} \cdot f'(x)}{1} =$$

$f(g(x))'$

$$= \lim_{x \to 0} \frac{f'(x)}{f(x)} = \lim_{x \to 0} \frac{6^x \ln 6 - 3^x \ln 3 + 2^x \ln 2}{6^x - 3^x + 2^x} =$$

$$\boxed{\frac{d}{dx} a^x = a^x \ln a, \quad a > 1}$$

$$= \frac{6^0 \ln 6 - 3^0 \ln 3 + 2^0 \ln 2}{6^0 - 3^0 + 2^0} = \ln 6 - \ln 3 + \ln 2 =$$

$$= \ln\left(\frac{6}{3} \cdot 2\right) = \ln 4$$

$$\boxed{L = \ln 4 = 2 \ln 2}$$

③ $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\dfrac{e^x}{\sqrt{x^2+1}}$.

$$\lim_{x\to 0}\frac{f(x)-f(-x)}{x}=\,?$$

Soluție: Notăm cu L limita. $f(-x)=\dfrac{e^{-x}}{\sqrt{(-x)^2+1}}=\dfrac{e^{-x}}{\sqrt{x^2+1}}$

$$\frac{f(x)-f(-x)}{x}=\frac{\dfrac{e^x}{\sqrt{x^2+1}}-\dfrac{e^{-x}}{\sqrt{x^2+1}}}{x}=\frac{e^x-e^{-x}}{\sqrt{x^2+1}\;x}$$

$$\frac{\dfrac{e^x-e^{-x}}{\sqrt{x^2+1}}}{x}$$

$$\frac{\dfrac{e^x - e^{-x}}{\sqrt{x^2+1}}}{x} = \frac{e^x - e^{-x}}{\sqrt{x^2+1}} \cdot \frac{1}{x} = \boxed{\frac{e^x - e^{-x}}{x\sqrt{x^2+1}}} =$$

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$$

$$= \frac{e^x - \dfrac{1}{e^x}}{x\sqrt{x^2+1}} = \frac{\dfrac{e^{2x} - 1}{e^x}}{x\sqrt{x^2+1}} =$$

$$= \frac{e^{2x} - 1}{e^x} \cdot \frac{1}{x\sqrt{x^2+1}} = \boxed{\frac{e^{2x} - 1}{e^x \cdot x\sqrt{x^2+1}}} \rightsquigarrow \text{limită} = ?$$

$$L = 2\lim_{x \to 0} \frac{e^{2x}-1}{2x\, e^{x}\sqrt{x^2+1}} = 2\underbrace{\lim_{x\to 0}\frac{e^{2x}-1}{2x}}_{=1}\ \underbrace{\lim_{x\to 0}\frac{1}{e^{x}\sqrt{x^2+1}}}_{=1}$$

$$\boxed{L = 2}$$

4) $f : (1, +\infty) \to (0, +\infty)$, $f(x) = \ln(x+1) - \ln(x-1)$.

$$\lim_{x \to +\infty} \left(x f(x)\right) = ?$$

<u>Soluție</u>: Notăm cu L limita.

$$f(x) = \ln\left(\frac{x+1}{x-1}\right)$$

$$L = \lim_{x \to +\infty} x \ln\left(\frac{x+1}{x-1}\right) = \lim_{x \to +\infty} x \ln \frac{1+\frac{1}{x}}{1-\frac{1}{x}} \overset{!!}{\wedge} \quad \infty \cdot 0 \; ??$$

$\hookrightarrow$ reducem la cazul $\frac{0}{0}$ sau $\frac{\infty}{\infty}$

$$\frac{x+1}{x-1}$$

$$L = \lim_{x \to +\infty} \frac{\ln\left(\frac{x+1}{x-1}\right)}{\frac{1}{x}} \rightsquigarrow \text{putem aplica l'Hôpital (Metoda I)}$$

Metoda II:

$$L = \lim_{x \to +\infty} x \ln\left(\frac{x+1}{x-1}\right) ; \quad \frac{x+1}{x-1} = \frac{(x-1)+2}{x-1} = 1 + \frac{2}{x-1}$$

$$L = \lim_{x \to +\infty} x \underbrace{\ln\left(1 + \frac{2}{x-1}\right)}_{} =$$

↳ Transformăm în $\ln\left(1 + \frac{1}{n}\right)$

$$= \lim_{x \to +\infty} x \ln\left(1 + \frac{1}{\frac{x-1}{2}}\right) = \lim_{x \to +\infty} (x - 1 + 1) \ln\left(1 + \frac{1}{\frac{x-1}{2}}\right)$$

$$= 2 \underbrace{\lim_{x \to +\infty} \frac{x-1}{2} \ln\left(1 + \frac{1}{\frac{x-1}{2}}\right)} + \underbrace{1 \cdot \lim_{x \to +\infty} \ln\left(1 + \underbrace{\frac{1}{\frac{x-1}{2}}}_{\to 0}\right)}_{= 0} \Rightarrow$$

$$\frac{\text{Limita}}{\text{Fundamentala}} \; 1$$

$$\Rightarrow L = 2 \cdot 1 + 0 = 2 \Rightarrow \boxed{L = 2}$$

$$\lim_{n \to +\infty} n \ln\left(1 + \frac{1}{n}\right) = 1$$

$$\left(n = \frac{x-1}{2}\right)$$

⑤ $f:\mathbb{R}\to\mathbb{R},\ f(x)=(x-5)(x-4)(x-3)(x-2)+1$

$$\lim_{n\to+\infty}\left(\frac{f(n+1)-1}{f(n)-1}\right)^{n}=?$$

321; răsturnatul: 123
inversul: $\frac{1}{321}$

Soluție: Notăm cu L limita.

$$\frac{f(n+1)-1}{f(n)-1}=\frac{\cancel{(n-4)}\cancel{(n-3)}\cancel{(n-2)}(n-1)\cancel{+1}\cancel{-1}}{(n-5)\cancel{(n-4)}\cancel{(n-3)}\cancel{(n-2)}\cancel{+1}\cancel{-1}}=\frac{n-1}{n-5}$$

$$L=\lim_{n\to+\infty}\left(\frac{n-1}{n-5}\right)^{n}=\lim_{n\to+\infty}\left(\frac{\cancel{n}\left(1-\frac{1}{n}\right)}{\cancel{n}\left(1-\frac{5}{n}\right)}\right)^{n}=\lim_{n\to+\infty}\left(\frac{1-\frac{1}{n}}{1-\frac{5}{n}}\right)^{n}\quad 1^{\infty}\ !!\ ?$$

$$\left(\frac{n-1}{n-5}\right)^n = \left(\frac{\frac{n-1}{n}}{\frac{n-5}{n}}\right)^n = \left(\frac{n-1}{n}\cdot\frac{n}{n-5}\right)^n$$ IDEE

$$= \left(\frac{n-1}{n}\right)^n \cdot \left(\frac{n}{n-5}\right)^n$$

$$L = \lim_{n\to\infty} \boxed{\left(\frac{n-1}{n-5}\right)^n}$$

Prelucrăm

Recomandare: Ne gândim la limita

$$\boxed{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e}$$

$$\boxed{\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n = e^a,\ a\in\mathbb{N}^*}$$

$$\left(\frac{n-1}{n-5}\right)^n = \left(\frac{n-5+4}{n-5}\right)^n = \left(1+\frac{4}{n-5}\right)^n =$$

$$= \left(1+\frac{4}{n-5}\right)^{n-5} \cdot \left(1+\frac{4}{n-5}\right)^5$$; Trecând la limită,

$$L = \underbrace{\lim_{n\to+\infty}\left(1+\frac{\boxed{4}}{n-5}\right)^{n-5}}_{\text{Limită Fundamentală } e^4} \underbrace{\lim_{n\to+\infty}\left(1+\cancel{\frac{4}{n-5}}^{\,0}\right)^5}_{=1} = e\cdot 1 = e$$

$$\Rightarrow \boxed{L = e^4}$$

Continuitate funcțiilor cu acoladă

① Să arătăm că funcția $f: \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} x, & x \leq 1 \\ x^2, & x > 1 \end{cases}$$ este continuă.

Soluție: f continuă pe $\mathbb{R} \setminus \{1\}$; Continuitate în punctul $x = 1$?

Dacă vrem continuitate într-un punct, $\boxed{l_s = l_d = f(1)}$!

$l_s = \lim\limits_{x \to 1, x<1} f(x) = \lim\limits_{x \to 1, x<1} x = 1$; $l_d = \lim\limits_{x \to 1, x>1} f(x) = \lim\limits_{x \to 1, x>1} x^2 = 1$

$f(1) = 1$ $\Rightarrow$ f continuă în $x = 1$

Limite de funcții (cu integrale):

② $f:(0,+\infty)\to\mathbb{R}$, $f(x)=\dfrac{1}{x^4}\displaystyle\int_0^x t^2e^{-t^2}\sin t\,dt$.

$\lim\limits_{x\to 0} f(x)=?$

Integrare prin părți + Integrală Riemann

Soluție: Idee: $\boxed{\text{L'Hôpital}}$!!

$$\frac{d}{dx}\underbrace{\int_0^x \overbrace{t^2e^{-t^2}\sin t}^{=\,g(t)}\,dt}_{G(x)-G(0)}=\frac{d}{dx}\big(G(x)-G(0)\big)=\boxed{g(x)}$$

Teorema Leibniz - Newton

$$\int_a^b g(x)\,dx = G(b) - G(a) \;;\; G \to \text{o primitivă a funcției } g$$

Aplicăm L'Hôpital :

$$L = \lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\int_0^x t^2 e^{-t^2} \sin t \, dt}{x^4} = \lim_{x\to 0} \frac{\frac{d}{dx}\int_0^x \cdots dt}{4x^3} =$$

$$= \cancel{\frac{g(x)}{}} \lim_{x \to 0} \frac{g(x)}{4x^3} = \lim_{x \to 0} \frac{\cancel{x^2} e^{-x^2} \sin x}{4x^{\cancel{3}}} =$$

$$= \lim_{x \to 0} \frac{e^{-x^2} \sin x}{4x} = \underbrace{\lim_{x \to 0} \frac{e^{-x^2}}{4}}_{= \frac{1}{4}} \underbrace{\lim_{x \to 0} \frac{\sin x}{x}}_{= 1 \text{ (limită fundamentală)}} = \frac{1}{4} \cdot 1 = \boxed{\frac{1}{4}}$$

Idee: $\boxed{\frac{d}{dx} \int_0^x f(t)\,dt = f(x)}$

$\boxed{\text{Exercitii de antrenament}}$

1) $f:\mathbb{R}\to\mathbb{R}$, $f(x)=4x-\ln(x^2+1)$

$$\lim_{x\to+\infty}\left(f(x+1)-f(x)\right)=?$$

2) $g:(0,+\infty)\to\mathbb{R}$,

$$g(x)=\frac{x^2+4x+4}{(x+2)^2e^x};$$

$$\lim_{n\to+\infty}\left(g(1)+g(2)+\dots+g(n)\right)=?$$

3) $f:(0,+\infty)\to(0,1)$, $f(x)=\sqrt{x+1}-\sqrt{x}$

$$\lim_{n\to\infty}\left(\frac{3}{2}+f'(1)+f'(2)+\dots+f'(n)\right)^{\sqrt{n}}=?$$

4) $f:(-1,1)\cup(1,+\infty)\to\mathbb{R}$,

$$f(x)=\frac{1}{(x-1)^2}-\frac{1}{(x+1)^2}.$$

$$\lim_{n\to+\infty}\left(f(2)+f(4)+f(6)+\dots+f(2n)\right)^n=?$$

5) $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\sqrt{x^2+1}$;

$$\lim_{x\to0}\frac{1}{x^2}\int_0^x t\,f(t)\,dt=?$$

(6) $f:\mathbb{R}\to\mathbb{R},\ f(x)=1+\dfrac{1}{\sqrt{x^2+1}}$

$$\lim_{x\to 0}\frac{1}{x}\int_0^x f(t)\,dt=?$$

(7) $f:\mathbb{R}\to\mathbb{R},\ f(x)=\dfrac{\ln(x+2)}{x^2+1}$

$$\lim_{x\to 0}\frac{1}{x}\int_0^x f(t)\,dt=?$$

BONUS $f:\mathbb{R}\to\mathbb{R},\ f(x)=(x^2+2)e^{-x}$

$$\lim_{x\to 0}\frac{1}{x}\int_0^x f(t)\,dt=?$$

Soluție: Notăm cu L limita. Aplicăm L'Hôpital:

$$L=\lim_{x\to 0}\frac{\int_0^x f(t)\,dt}{x}=\lim_{x\to 0}\frac{\frac{d}{dx}\int_0^x f(t)\,dt}{1}=$$

$$=\lim_{x\to 0} f(x)=\lim_{x\to 0}(x^2+2)e^{-x}=2\underbrace{e^0}_{=1}=2$$

$$\boxed{L=2}$$